A Combined Geometrical and Topological Simplification Hierarchy for Terrain Analysis

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ABSTRACT

We consider the problem of modeling a terrain from both a geometric and a morphological point of view for efficient and effective terrain analysis on large data sets. We devise and implement a simplification hierarchy for a triangulated terrain, where the terrain is represented as a triangle mesh and its morphology is described by a discrete Morse gradient field defined on the basis on the elevation values given at the vertices of the mesh. The discrete Morse gradient is attached to the triangles, edges and vertices of the mesh. We define a new edge-contraction operator for the edges of the triangle mesh, which does not change the behavior of the gradient flow and does not create new critical points, and we apply it to the original full-resolution mesh in combination with a topological simplification operator which eliminates critical simplices in pair. We build the simplification hierarchy based on suitably combining such operators and we evaluate it experimentally.

Categories and Subject Descriptors

E.1 [**Data**]: Data Structures—*Graphs and networks*; I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling; I.3.6 [**Computer Graphics**]: Methodology and Techniques—*Graphics data structures and data types*

Keywords

Terrain Modeling, Morphological Analysis, Discrete Morse Theory

1. INTRODUCTION

Morse complexes and Morse-Smale (MS) complexes, rooted in Morse theory, have gained much interest as powerful tools for providing a structural description of a terrain which constitutes the basis for analysis, visualization and semantic annotation. Forman [7] has developed a discrete analogue of Morse theory for cell and simplicial complexes. Since this approach is entirely combinatorial, it avoids computing derivatives and is beneficial in the presence of noise in the data, and thus is a good choice for dealing with discrete data. The huge size of available datasets poses a variety of problems also for their topological representation. In the literature, simplification techniques for a terrain at different topological resolutions have been proposed. Simplification of Morse and Morse-Smale complexes can be achieved by applying an operator, called *cancellation* [10], which consists of removing two critical points of consecutive index (i.e., a maximum and a saddle, or a saddle and a minimum) which are connected by a separatrix line. This operator has been investigated for 2D [6, 3] and 3D [9, 4] Morse-Smale complexes. Issues arise in the geometrical representation associated with the simplified Morse or Morse-Smale complexes. Current approaches either maintain the original triangle mesh at full resolution as geometrical description associated with the cells of the Morse or MS complex, or they approximate from scratch the two-dimensional cells of the MS complexes [2, 3, 12]. Hierarchical models for Morse or MS complexes have been developed for terrain modeling based on simplification hierarchies [6, 3, 5].

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Here, we define and implement the first simplification operator for triangle mesh endowed with a Forman gradient, a gradient-aware edge-contraction, which avoids deleting or creating critical simplices, thus maintaining the gradient behavior at each update. By interleaving sequences of gradient-aware edge-contraction, to reduce the size of the triangle mesh, and of topological simplifications, to reduce the number of critical points of the terrain, we are able to build up a simplification hierarchy for both terrain geometry and morphology. Such simplification hierarchy is at the basis for a combined geometric and topological multi-resolution model which would allow extractions of adaptive terrain representations at variable geometrical and topological resolutions.

2. DISCRETE MORSE THEORY

Morse theory [10] studies the relationships between the topology of a manifold M and the critical points of a scalar function f defined on it. In the literature, two extensions of Morse theory to a discrete domain can be found, namely *piecewise-linear Morse theory* [1] and *discrete Morse theory* [7]. Discrete Morse theory is a discrete counterpart of Morse theory for simplicial and cell complexes. In this case, a discrete Morse function is defined on all the cells of the complex. For the sake of simplicity, we will briefly review this theory for 2D simplicial complexes, i.e., triangle meshes. The vertices, edges and triangles of a triangle mesh are collectively called *simplices*. A vertex is a 0-simplex, an edge a 1-simplex and a triangle a 2-simplex.

A function $F : \Sigma \to \mathbb{R}$, defined on a triangle mesh Σ, is a *discrete Morse function* (also called a *Forman function*) if for every i-simplex $\sigma \in \Sigma$, all the $(i-1)$-simplices on the boundary of σ have a lower function value than σ, and all the $(i+1)$-simplices bounded by σ have a higher function value than σ, with at most one exception. If there is such an exception, it defines a pairing of cells, called a

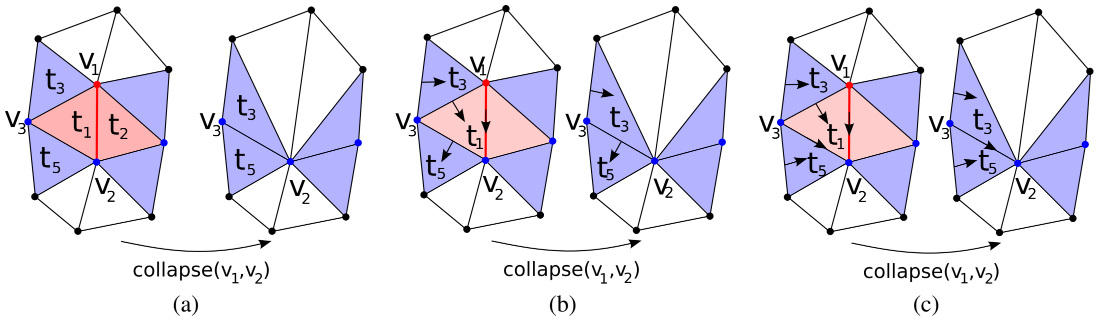


Figure 1: (a) Triangles and edges involved in an edge-contraction (edge e is depicted in red). From left to right we show the result of the contraction of edge e. In (b) and (c) two different gradient configurations before and after an edge-contraction are shown.

gradient pair. Otherwise, σ is a *critical* simplex of index p. Thus, an i-simplex σ is not critical if and only if there exists an $(i-1)$-simplex τ such that $F(\tau) \geq F(\sigma)$ or an $(i-1)$-simplex β such that $F(\beta) \leq F(\sigma)$. These two cases are mutually exclusive, i.e., a simplex σ can be paired either with a non-critical simplex bounded by σ or with one of its faces. A pair can be viewed as an *arrow* formed by a head (i-simplex) and a tail ($(i-1)$-simplex). A simplex that is not a head or a tail of any arrow is a *critical* simplex.

A V*-path* is a sequence of simplices $[\sigma_0, \tau_0, \sigma_1, \tau_1, ..., \sigma_i, \tau_i, ..., \sigma_q, \tau_q]$ such that σ_i and σ_{i+1} are on the boundary of τ_i and (σ_i, τ_i) are paired simplices, where $i = 0, ..., q$. The collection of all paired and critical simplices of Σ forms a *discrete Morse gradient* (also called a *Forman gradient*) V if all the V-paths are acyclic. V-paths correspond to the integral lines of the discrete Morse function F defined on Σ. We will call *separatrix* V_j*-path* any V-path of the following form: $[\tau, \sigma_0, \tau_0, \sigma_1, \tau_1, ..., \sigma_i, \tau_i, ..., \sigma_q, \tau_q, \sigma]$, where τ and σ are two critical simplices of dimension j and $(j-1)$, respectively.

In this work, we consider a discrete terrain model as a pair (Σ,f), where Σ is the triangle mesh discretizing the domain, and f is the elevation given at the vertices of Σ, which is the information we are given as input, and we build from Σ a discrete Morse gradient field V compatible with the elevation function f. Σ endowed with gradient field V is denoted as (Σ,V).

3. A GRADIENT-AWARE MESH SIMPLIFICATION OPERATOR

The most common operator for simplifying triangle meshes is *edge contraction*. An *edge-contraction* acts on mesh Σ by contracting an edge e, with endpoints v_1 and v_2, to one of its endpoints (i.e., v_2) (see Figure 1(a)). When mesh Σ is endowed with a discrete Morse gradient V, we have to modify also the pairings in V accordingly. The key idea here is to locally modify V without modifying the critical vertices, edges or triangles, or adding new critical simplices, thus maintaining the behavior of the gradient flow. We impose feasibility conditions on mesh Σ and on the discrete Morse gradient, i.e., on the pair (Σ, V).

An edge-contraction, $contract(v_1, v_2)$, is feasible on mesh (Σ, V) if and only if: (i) all the simplices to be removed (edge e, vertex v_1 and triangles t_1 and t_2) or to be modified (edges and triangles having vertex v_1 on their boundary) by the edge contraction are not critical; (ii) v_1 is paired with edge e in V; (iii) there are at least three triangles in Σ incident in v_1.

Clearly, no critical simplex is removed during simplification, because of condition (i). We need to show that we do not introduce any critical simplex. Condition (iii) is required to avoid having a triangle incident in v_1 and adjacent to both t_1 and t_2, since this would not guarantee that we do not introduce new critical simplices. Then, we will show that all the simplices paired before the application of $contract(v_1, v_2)$ are still paired after, and this guarantees that no critical simplex is introduced. We show this for the left part of the star (which is defined by the set of triangles incident in e and in its extreme vertices) of the oriented edge e to be contracted only, since the updates required are symmetrical on the left and on the right part. We denote as v_3 the vertex of t_1 different from v_1 and v_2, and we denote as t_3 and t_5 the triangles adjacent to t_1 on the edge opposite to v_2 and v_1, respectively. Specifically, for the left part, vertex v_3, edge (v_3, v_1) and triangles t_3 and t_5 must be paired before and after the simplification.

The updates on V depend on edges (v_3, v_2) and (v_3, v_1). If these edges are both paired with a triangle (i.e., see Figure 1(b)), then t_3, (v_3, v_2) and t_5 will have the same pairs after the simplification (see Figure 1(b)). The same holds when the gradient arrows have opposite direction (i.e., (v_3, v_2) is paired with t_1 and (v_3, v_1) is paired with t_3). When (v_3, v_2) is paired with one of its vertices (see Figure 1(c)), t_3 is necessarily paired with (v_3, v_1), since t_3 cannot be critical). Thus, the removal of both (v_3, v_1) and t_3 does not make any paired simplex unpaired (see Figure 1(c)). When (v_3, v_1) is paired with one of its vertices we get the same result.

4. FORMAN GRADIENT SIMPLIFICATION OPERATORS

We use two simplification operators to modify the morphology of the terrain by operating on the Forman gradient V defined on Σ. Such operators are called *cancellation*, and extend to discrete Morse theory the cancellation operators defined in the smooth case [10]. Operator $1-cancellation$ applied to (Σ, V) deletes a critical edge e and a critical triangle t connected through a separatrix V_2-path. If we denote as t' the other critical triangle connected to e through a separatrix V_2-path, the effect of a $1-cancellation(e, t, t')$ on V is to delete t and e from the sets of critical simplices of V and to reverse the gradient arrows on the only separatrix V_2-path from t to e (see Figure 2).

Dually, operator $0-cancellation$ applied to (Σ, V) deletes a critical vertex v and a critical edge e connected through one separatrix V_1-path. We call v' the other critical vertex connected to e through a separatrix V_1-path. The effect of a $0-cancellation(e, v, v')$ on V is to delete v and e from the sets of critical simplices of V and to reverse the gradient arrows on the only separatrix V_1-path between

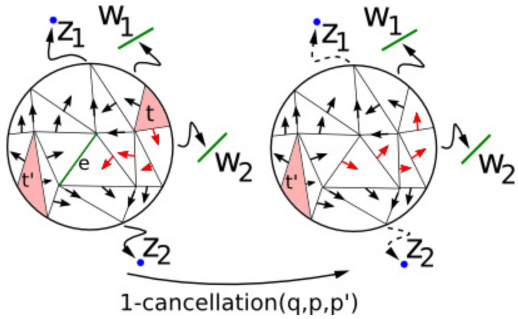


Figure 2: Effect of 1-$cancellation(e,t,t')$ **on a Forman gradient** V **defined on a triangle mesh. The original gradient field** V**, on the left side, with two critical triangles** t **and** t' **(red triangles) and one critical edge** e **(green edge). Separatrix** V**-paths, outside the portion of the triangle mesh shown, are depicted with bold lines. Regular** V**-paths, outside the portion of the triangle mesh shown, are depicted with dotted lines. Red arrows indicate the** V**-path involved in the simplification.**

v and e.

5. GRADIENT-AWARE SIMPLIFICATION HIERARCHY

We are interested in extracting representations of a terrain at different resolutions, for both the underlying triangle mesh, and the morphology. To this aim, we have combined the geometric and topological simplification operators described in Sections 3 and 4. We generate a simplification hierarchy from a sequence of gradient-aware *edge-contraction* operators, reducing the size of the triangle mesh, and from a sequence of topological *cancellation* operators, simplifying the morphology of the terrain.

The discrete terrain model given as input consists of a triangle mesh Σ_{full} discretizing the domain, and a function f giving the elevation values at the vertices of Σ_{full}. The first step consists of computing a discrete Morse gradient, the Forman gradient V associated with the simplices of Σ_{full} and compatible with function f. Recall that gradient field V consists of a collection of critical simplices plus a collection of pairs of the type (triangle, edge) or (edge, vertex). To compute V, we have adapted the algorithm by Robins et al. [11] from regular grids to triangle meshes.

Starting from pair (Σ_{full}, V) the simplification hierarchy is built by an alternating sequence of geometric and topological simplifications. As a first step, all the feasible edge-contractions are performed in increasing order of edge length. Note that we use edge length for simplicity just to prove the feasibility of the approach. To obtain a higher fidelity to the original mesh, we plan to experiment with other quality measures, such as the Quadric Error Metrics (*QEM*) [8]. We perform, at each simplification step, the maximum number of possible independent simplifications. When all feasible edge-contractions have been performed, we construct a queue, with the topological simplifications, in increasing order of *persistence* and we perform all of them. The persistence of an i-cancellation involving two critical simplices is defined as the difference in absolute value of the elevations at the corresponding critical points (maxima, minima or saddles) in the original triangulated terrain [6]. The interleaving between collections of geometric and of topological simplifications continues in the simplification process until no more simplification is possible. The resulting mesh, that we call the *base mesh* and denote as Σ_B, is endowed with a simplified Forman gradient, that we denote as V_B.

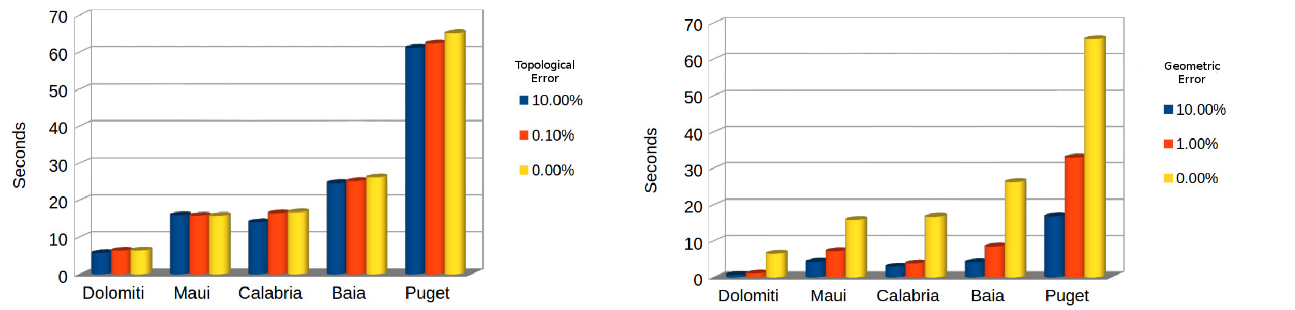
6. EXPERIMENTAL RESULTS

The purpose of our experiments is to show the efficiency of our approach in computing the Morse complexes on representations of the triangulated terrain at different level of resolutions. Experiments have been performed on a desktop computer with a 3.2Ghz processor and 16GB of memory. The size of the triangle meshes used is between 1.6M and 19M triangles. By analyzing the number and kind of simplifications performed, we have noticed that the geometrical simplifications account for almost 90% of the total number of simplifications. This behavior emphasizes the importance of simplifying the geometry when we are interested in topological analysis.

The main purpose of our simplification hierarchy is to allow an efficient topological inspection of a terrain dataset. To this aim, we consider the discrete ascending and descending Morse complexes. Since the time complexity for extracting the descending/ascending Morse complex depends on the total number of simplices (vertices, edges and triangles) and not on the number of critical simplices, simplifying just the topology will not lead to a reduction in the time required for extracting the Morse cells. To verify such behavior, we have computed the cells of the Morse complexes on several representations simplified at different levels of topological resolution. An input parameter sets the persistence threshold, for the topological resolution, computed as a percentage of the total elevation range. In Figure 3(a), we illustrate the timings for computing all the cells of Morse complexes from the representations obtained. We notice that, independently of the simplification error for the topology, having the underlying geometry at full resolution leads to the same timings for computing the Morse complexes. On the contrary, as shown in Figure 3(b), varying the resolution of the underlying geometry can affect the computation of the Morse cells considerably. In the experiments we have performed extractions at full resolution for the topology of each terrain data sets (i.e., without any topological simplification) simplifying the geometry with the error indicated and, as a result, the computation of the Morse cells is 2 to 6 times faster than computing them on the mesh at full resolution. In Figure 4 we show two examples of a critical net (i.e. ascending and descending 1-cells) computed on two representations of the Dolomiti dataset. Red points correspond to maxima, green points to saddles and blue points indicate minima. Descending 1-cells (depicted in blue) are chains of edges starting from a saddle and ending into a minimum. Ascending 1-cells (depicted in red) are chains of triangles starting from a maximum and ending into a saddle. In Figure 4(a) we strongly reduce the topological resolution (8K simplifications) but we leave geometry at full resolution, in Figure 4(b), instead, we use the same topological resolution on the mesh at coarsest resolution. We have observed that the geometry resolution directly affects the computation of Morse features and we could be tempted to use only the coarsest geometrical representation. However (see Figure 4(b)), Morse features are generally applied to scalar field analysis for analyzing and/or visualizing relevant parts of the terrain and, for such goals, the coarsest representation is generally insufficient. For this reason a variable-resolution model would offer a good solution for augmenting the geometric resolution only in specific regions of interest.

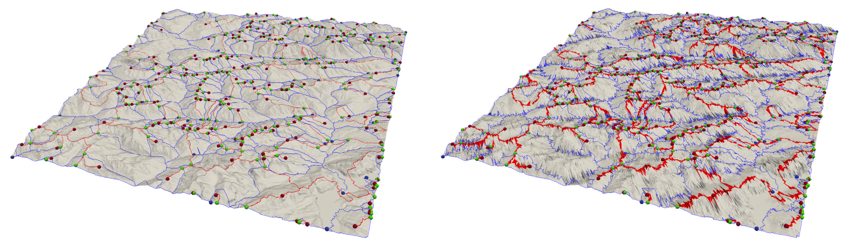
7. CONCLUDING REMARKS

The simplification hierarchy proposed here is a first step towards the definition of a multi-resolution model for the interactive exploration



(a) (b)

Figure 3: (a) Time required for computing Morse complexes on simplified representations with various topological resolutions and with geometry at full resolution. (b)Time required for computing Morse complexes on representations with geometry and topology at full resolution.



(a) (b)

Figure 4: Representations for the Dolomiti dataset obtained varying the topological resolution (8K simplifications applied) and with geometry at (a) full and (b) coarsest resolution.

of a terrain, both from a geometric and morphological points of view. Our current plan is to define gradient-aware edge-contraction for tetrahedral meshes endowed with a Forman gradient and develop a Forman hierarchy for 3D scalar fields defined on unstructured tetrahedral meshes for volume data analysis and visualization. Since both the edge collapse and the simplification of a Forman gradient are operations defined in a dimension independent way we are going towards the definition of a multi-resolution model for scalar fields defined on volume meshes.

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